

Renormalization of coherent state variables, within the geometrical mapping of algebraic models

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We investigate the geometrical mapping of algebraic models. As particular examples we consider the *Semimicroscopic Algebraic Cluster Model* (SACM) and the *Phenomenological Algebraic Cluster Model* (PACM), which also contains the vibron model as a special case. In the geometrical mapping coherent states are employed as trial states. We show that the coherent state variables have to be renormalized and not the interaction terms of the Hamiltonian, as is usually done. The coherent state variables will depend on the total number of bosons and the coherent state variables. The nature of these variables is extracted through a relation obtained by comparing physical observables, such as the distance between the clusters or the quadrupole deformation of the nucleus, to their algebraic counterpart.

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I. INTRODUCTION

Algebraic models play an important role in nuclear and particle physics in understanding collective properties of nuclei [1], clusters of nuclei [2–6], atomic molecules [7] and the spectrum of hadrons [8], just to name a few examples. In order to achieve a more physically inspired interpretation it is useful to employ a geometrical mapping, using coherent states [9–11]. More recent investigations on geometrical mappings can be found in Refs. [12–15]. The semi-classical potential is determined by the expectation value of the Hamiltonian with respect to the normalized coherent state. To the best of our knowledge Ref. [16] was the first work in which coherent states were applied within an algebraic model, namely the *Interacting Boson Approximation* (IBA).

In establishing a relation with geometric quantities the following procedure is usually applied: Denoting generically the coherent state variables by α , the expectation value of the one-, two- and three-body interactions with respect to the coherent state is obtained as $N\alpha^2$, $N(N-1)\alpha^4$ and $N(N-1)(N-2)\alpha^6$, respectively [9, 10, 12, 13, 16]. The α variable is then interpreted to be proportional (in lowest order) to a physical variable, for example the deformation β of a nucleus [1] or some other coordinate [9, 12, 13]. The dependence on N is used as an argument to renormalize the different order of interactions [9, 10, 12, 13]. Apparently the first appli-

cation of this procedure to atomic and nuclear systems was in Refs. [9, 10], where the expectation values, with respect to the coherent state, of the one- and two-body interactions in the vibron model were divided by N and $N(N-1)$ respectively, with the argument that "it is convenient" to define an intensive \mathbf{H} . To us, this argument is not conclusive nor justified, on grounds that intensive quantities are defined by dividing an operator globally by N , rather than dividing individual terms in an expansion independently by different powers of N . The expectation values, divided usually by appropriate powers in N , were set equal to α^2 and α^4 , which in turn were assumed to be of the order of one and identified with physical coordinates, such as the relative distance between two mass points. (The momentum dependence was also discussed, using complex coherent state variables. For simplicity, in order to describe the problem and the solution to it, we are only interested in the coordinate dependence and will set the imaginary part of the cartesian components of α equal to zero.) Later on a general rule became accepted according to which one-, two- and three-body interactions are divided by N , $N(N-1)$ and $N(N-1)(N-2)$, respectively. This dependence was also suggested in [16] in the early years of the IBA.

Reference [17] gives a clearer insight into why the former procedure is applied. We suspect that the origin of the renormalization of the interaction parameters is found in chapter 15 of [17], where systems of N identical

bosons or fermions are discussed, and thus we provide here a summary of material therein.

The Hamiltonian is a function of the generators of a Lie algebra (see also [18]), i.e. $\mathbf{H}(\mathbf{T}_i)$, with \mathbf{T}_i as short-hand notation for the generators. The thermodynamical limit of a Lie algebra is obtained by dividing each generator by N (i.e., one defines \mathbf{T}_i/N) giving a new generator. The newly defined generators commute in the limit $N \rightarrow \infty$ and thus represent classical operators. This limiting procedure corresponds to a *contraction* of a Lie algebra and is well described in Ref. [19]. A model Hamiltonian is then given as a function in \mathbf{T}_i/N and additionally multiplied by N . The Hamiltonian has the structure $\mathbf{H} = N\mathbf{h}(\mathbf{T}_i/N)$, and thus a scaling of the Hamiltonian in N is *additionally* imposed. As a special example, the Meshkov-Glick-Lipkin (MGL) and the Dicke model are discussed. Factors in powers of $1/N$ are introduced in order to maintain the contribution of a higher order interaction on the same level as that of a one-body interaction. The scaling of the interaction terms is necessary, because in a two-level model the lowest level is completely occupied and when the number of states in the lowest level is increased, so is N . Without this renormalization of the interaction parameters the two-body interaction would completely dominate the one-body interaction, resulting a useless model. In other words, the proposed scaling recipe is introduced by necessity.

A further argument Ref. [19] provides is that in nuclear physics a clear scaling of the binding energy is observed and it should be reproduced in realistic models. As logical as this arguments sounds, in models of nuclear physics, like the shell model, one does not always have a scaling in the interaction terms as a function in the number of nucleons. For example, in order to obtain the scaling of the binding energy, the *Liquid Drop Model* [20] is applied. To incorporate corrections due to the shell structure, the Strutinsky method is used (see, for example, [21]). The average behavior of the calculated energy, as a function on the number of nucleons, is subtracted and the result, containing the fluctuations due to the shell structure, is then added to the Liquid Drop Model. Thus, in the shell model itself no scaling of the Hamiltonian appears.

Another example is the geometric model of the nucleus [20, 22]. The basic degrees of freedom are spin two bosons, related to the surface vibration and rotation of the nucleus. As in the IBA [1] one can introduce a spin zero boson and impose the condition that the total number of bosons N is fixed. The cutoff is removed by letting N go towards large values. In this geometric model no rescaling of the interaction terms in inverse powers in N is applied, because this would annihilate the higher n -body interaction compared to the one-body interaction, thus eliminating important structures.

This shows that a model Hamiltonian does not necessarily require a renormalization of the interaction parameters. As we shall show further below, the interaction parameters of the Hamiltonians, discussed in this

contribution, will depend on a cutoff and approach fixed values for a large cutoff. It shows that the rules given in Refs. [17, 18] are not always valid and depend on the model itself. Applying these rules indiscriminately may lead to inconsistent results, though the rules are justified with regards to certain types of models. As we will see, inconsistencies become particular noticeable for models where the N has to approach infinity (removing a cut-off). In section II we first summarize the definitive features of all algebraic Hamiltonian methods, as well as some problems in the conventionally-used renormalization procedure used for such. In section III we discuss the geometric mapping. In section IV we propose an alternative procedure that is based on the relation of algebraic operators to their geometric counterparts. This will relate the coherent state parameters α to a physical quantity. These variables will depend on N and are approximately proportional to the given physical quantity. The N -dependent factors in front of an n -body interaction will disappear in the large N limit, so no renormalization of the n -body interaction will be required. Instead a renormalization of the coherent state variables, α , becomes necessary. We have the impression that the reason for the confusion is related to treating the N -dependent factors and the coherent state variables (α) as of different origin and therefore independent. However, they are not. As the main examples of an algebraic model, we shall consider the Semimicroscopic Algebraic Cluster Model (SACM) and the Phenomenologic Algebraic Cluster Model (PACM). After discussing the results in section V, conclusions are drawn in section VI.

II. THE STRUCTURE OF ALGEBRAIC HAMILTONIANS

An algebraic Hamiltonian of boson systems, which includes, in general, different types of bosons, exhibits the following dependence

$$\mathbf{H} = \mathbf{H}(\mathbf{a}^\dagger, \mathbf{a}, \mathbf{b}^\dagger, \mathbf{b}, \dots, p) \quad , \quad (1)$$

where the $\mathbf{a}_{i_a}^\dagger$, \mathbf{a}_{i_a} , $\mathbf{b}_{i_b}^\dagger$, \mathbf{b}_{i_b} , etc., are boson creation and annihilation operators, satisfying the usual commutation relations. The indices i_a, i_b, \dots vary from 1 to d_a, d_b, \dots , denoting the degrees of freedom for each type of boson. We associate a definite angular momentum l_a, l_b , etc. to each boson. Thus the total number of degrees of freedom is $d_a + d_b + \dots = d$, where $d_k = (2l_k + 1)$ ($k = a, b, \dots$). The p in (1) is a short-hand notation for the interaction parameters (p_k). The Hamiltonian is constructed as an $SO(3)$ scalar, i.e., all operators are coupled to angular momentum zero. These are examples where there are different kinds of particles, as opposed to those in Refs. [17, 18].

The interactions are usually divided into one-, two-, three- body and higher terms. The one-body interac-

tions, coupled to angular momentum zero, are given by

$$\mathbf{n}_a = \sum_{i_a} \mathbf{a}_{i_a}^\dagger \mathbf{a}_{i_a}, \mathbf{n}_b = \sum_{i_b} \mathbf{b}_{i_b}^\dagger \mathbf{b}_{i_b}, \dots \quad (2)$$

These are the respective number operators. In algebraic boson models, one requires that the *total number of bosons is constant*, i.e.,

$$N = \mathbf{n}_a + \mathbf{n}_b + \dots = \text{const} \quad (3)$$

A general one-body interaction is then given by

$$\sum_k \epsilon_k \mathbf{n}_k \quad (4)$$

Similar considerations can be given for higher-order interactions.

In Ref. [13], which gives a comprehensive resumé on the until-now accepted procedure, the different kinds of bosons are denoted by the same letter \mathbf{b} , but with indices that distinguish between different kinds of bosons. At least one boson has angular momentum zero, which we also shall adopt and identify the a -boson with this scalar boson. This implies that $i_a=1$, so this index can be skipped, for convenience.

The notation set up above is generic, and it contains all the special cases. For example, the IBA [1], which describes quadrupole excitations of the nucleus in its basic version, has two types of bosons, namely a scalar s -boson and a quadrupole d -boson. The total number of bosons, $N = n_s + n_d$, is given by half the number of nucleons in the valence shell and is therefore constant and *finite*.

The notation is also valid for the Semimicroscopic Algebraic Cluster Model (SACM) [3, 4] and the Phenomenologic Algebraic Cluster Model (PACM), which contains the *Vibron Model* as a special case, i.e. that in which the clusters are closed-shell nuclei [23, 24]. The former observes the Pauli exclusion principle between all the nucleons, including those belonging to different clusters, while the latter does not. In Refs. [23, 24] this difference was investigated and the geometric mapping was also performed in order to study quantum phase transitions. The basic degrees of freedom in the SACM and PACM are the relative oscillation quanta, described by spin one π -bosons. These bosons are the shell model oscillation quanta. In order that the algebraic Hamiltonian conserves the total number of bosons, a scalar σ -boson is introduced, which has no physical meaning; rather, its role is merely to introduce a cutoff. This cutoff has to tend to infinity, which, in general, requires $n_\pi \ll n_\sigma$. In some dynamical symmetry limits ($SO(4)$ in the PACM), the numbers may be of the same order. The PACM is also applied to molecules, where the Pauli exclusion principle is not required.

The main point here is that there are different kinds of bosons, which carry different angular momentum. They have different number of degrees of freedom, depending on the model and system. The number of bosons of a certain type, contained in a given eigenstate of the Hamiltonian, may be quite different from the number of other

type bosons. Coming back to the one-body interactions of Eq. (2), they are proportional to the respective boson numbers, i.e.,

$$\mathbf{n}_a \rightarrow n_a, \mathbf{n}_b \rightarrow n_b, \dots \quad (5)$$

and *not* to N as suggested, for example, in [13].

In the $U(5)$ dynamical symmetry limit of the IBA, the number of s - and d -bosons is well defined but, in general quite different. The number operator of these bosons is proportional to n_s and n_d , respectively. The corresponding number operators of the SACM and PACM are proportional to n_σ and n_π , respectively. Furthermore, because the total number of bosons has to be large and in general, $n_\pi \ll N$, i.e. $n_\sigma \gg n_\pi$, the matrix elements of the corresponding number operators are very different from those suggested in Ref. [13].

For the two-body interactions (this is the last example discussed and it should be clear how higher-order interactions behave), the matrix elements of interactions, which depend only on the a -bosons and b -bosons, are proportional to $n_a(n_a - 1)$ and $n_b(n_b - 1)$, etc., respectively. It is not difficult to imagine that two-body interactions, which depend on two types of boson, behave as $n_a n_b$, etc. Again, using coherent states they do not scale as $N(N-1)$, as suggested in [12], but rather as $N(N-1)\alpha^4$, i.e., the N and α dependence are linked together and in the scaling consideration one has to treat both at the same time.

The question is: How can the inconsistencies be resolved? In order to propose a possibility, we have to discuss coherent states used for geometrical mappings. After that, we shall analyze the origin of the inconsistencies of the old renormalization procedure and discuss how it should be modified to make it consistent.

III. THE GEOMETRICAL MAPPING USING COHERENT STATES

The general structure of a coherent state, used as a trial state in an algebraic model, is as follows:

$$|\alpha, \beta, \dots\rangle = \mathcal{N} (\alpha a^\dagger + (\beta \cdot b^\dagger) + \dots)^N |0\rangle \quad (6)$$

where \mathcal{N} is an easy to determine normalization factor, and the vacuum is denoted by $|0\rangle$. We assumed that the a -boson is the scalar one and the dot indicates a scalar product. The α can be factorized out, as done in some models.

Examples of coherent states are:

$$|\beta\rangle = \frac{1}{\sqrt{N!(1+\beta^2)}} [s^\dagger + (\beta \cdot d^\dagger)]^N |0\rangle \quad (7)$$

for the IBA and

$$\begin{aligned} |\alpha\rangle &= \mathcal{N}_{N,n_0} (\alpha \cdot \pi^\dagger)^{n_0} [\sigma^\dagger + (\alpha \cdot \pi^\dagger)]^N |0\rangle \\ &= \mathcal{N}_{N,n_0} \frac{N!}{(N+n_0)!} \frac{d^{n_0}}{d\gamma_1^{n_0}} [\sigma^\dagger + \gamma_1 (\alpha \cdot \pi^\dagger)]^{N+n_0} |0\rangle \end{aligned} \quad (8)$$

for the SACM, where following Refs. [23] we redefined the total number of relative oscillation quanta as $(N + n_0)$, while the γ_1 parameter has to be set equal to 1 after the differentiation. Here n_0 denotes the minimal number of π bosons as required by the Pauli principle. In the PACM $n_0 = 0$ and the coherent state reduces to that of the more commonly known vibron model [25].

In the IBA a relation between the β deformation variable and the one of the geometric model [21] is given in Ref. [26]. This relation is based on comparing the quadrupole density obtained in the two models. As a result, the physical quadrupole deformation, β , turns out to be roughly proportional to $(\frac{2N}{A})\beta$, where N is the total number of bosons and A is the mass number of a nucleus. These values are much lower than β . This demonstrates that the naive interpretation of β being identical to the value of the quadrupole deformation is incorrect and the correct one can only be found by comparing the expectation value of one operator in the geometric model to the equivalent one in the algebraic model. Additionally, as already mentioned, the identification of the geometric variable β_{phys} has been obtained for a given N together with the coherent state variable β , and not independently.

For further illustration, let us consider the expectation value of the π -boson number operator in the PACM, i.e. with $n_0 = 0$. (This quantity does not depend on an interaction parameter.) The result is [27]

$$\langle n_\pi \rangle = N \frac{\alpha^2}{(1 + \alpha^2)} . \quad (9)$$

We know that this expectation value is equal to the average number of π -bosons, which in the $SU(3)$ dynamical symmetry limit is n_π . Thus, it cannot be of the order of N , but rather

$$N \frac{\alpha^2}{(1 + \alpha^2)} = n_\pi , \quad (10)$$

which is of the order $n_\pi \ll N$. This suggests that the coherent state variable α has to be renormalized, as already indicated in the last section.

As just indicated, in order to obtain a consistent relation of the coherent state variables to physical geometrical variables, one has to rely on the comparison of the expectation value of one (or more) *algebraic* operator(s), with respect to a trial state (here a coherent state) to the corresponding *geometric* expression of the operator(s). This comparison gives a relation of the coherent state parameter with the corresponding geometrical variable.

IV. RENORMALIZATION OF THE COHERENT STATE VARIABLE α

Here we give a more detailed discussion of the ideas outlined above for the SACM and the PACM. The corresponding treatment of the IBA can be found in Ref.

[26], and the ideas can be transferred to other algebraic models.

The basic conditions we require for the Hamiltonian and the coherent state variables are that for large N

- i) the Hamiltonian cannot depend on an arbitrary cut-off, represented by the total number of bosons N , at least in the limit for very large N , and
- ii) the parameter α of the coherent state has to be related to an independent operator, as for example the distance operator.

Condition i) introduces the following requirement for the N -dependence of the interaction parameters in the Hamiltonian. Let $p(n)$ symbolize the parameter of an n -body interaction. In the SACM this parameter, once it is adjusted to experiment, should acquire a fixed value $p_0(n)$ for $N \rightarrow \infty$. When the cutoff N is chosen to be finite, the interaction parameter has a different value in general. The dependence on N can be parametrized in powers of $1/N$, i.e.,

$$p(n) = p_0(n) + \frac{p_1(n)}{N} + \frac{p_2(n)}{N^2} + \dots \quad (11)$$

Applying an adjustment to experimental data, with increasing N , the parameter $p(n)$ approaches $p_0(n)$. No renormalization of the interaction is required. In the IBA, the "renormalization", given by $p(2)/[N(N-1)]$ is equivalent to writing the parameter $p'(n)$ instead, because N is finite and fixed. This implies that it is a matter of convenience to include the N -dependent factor or not. This is quite different in the cluster models, where N is *arbitrary* but fixed. When the cutoff is removed, N has to tend to infinity. This difference is the main reason we insist on the example of the cluster models.

With respect to ii), we first define an algebraic distance operator, which for $N \rightarrow \infty$ approaches the usual distance operator (this discussion can also be found in Ref. [27]):

$$r_m = \sqrt{\frac{\hbar}{2m\omega}} (\pi_m^\dagger + \pi_m) . \quad (12)$$

When the σ -bosons are introduced, this operator has to be changed. The new operator, called the *algebraic coordinate operator*, should satisfy the following minimal conditions:

- a) The total number of bosons has to be kept constant, i.e., each π_m^\dagger has to be multiplied by σ and each π_m has to be multiplied by a σ^\dagger ;
- b) the definition of the distance operator should be *independent* of the basis and Hamiltonian used (though, it is permissible to give the representation of the operator in one particular basis); and
- c) for $N \rightarrow \infty$ it should converge to the standard form given in (12).

The proposed algebraic distance operator is given by

$$r_m^a = \sqrt{\frac{\hbar}{2Nm\omega}} (\pi_m^\dagger \sigma + \sigma^\dagger \pi_m) , \quad (13)$$

where “ a ” refers to *algebraic*. The operator itself leaves the total number of bosons unchanged, as required by condition a). The N in the denominator of the square root is introduced because in the harmonic oscillator basis the matrix elements of the σ -operators are proportional to $\sqrt{N - n_\pi}$, which for large N and small number of π bosons is approximated by \sqrt{N} . *This approximate value of the σ operators is satisfied in any basis, with the condition that the average number of π bosons is much smaller than N (though the structure is particularly simple in the harmonic oscillator basis).* Thus the $1/\sqrt{N}$ factor cancels approximately the contributions due to the addition of the σ^\dagger and σ operators. In this form, the algebraic distance operator does not depend on the basis used (\mathbf{r}_m^a can be applied to any kind of basis), nor on the Hamiltonian or any particular dynamical symmetry, thus satisfying condition b). For very large N the expressions of the physical and the algebraic coordinate operators tend to each other, satisfying condition c).

This definition agrees with [28, 29], where an algebraic model for *atomic molecules* is discussed. Often (see for example [30]) one defines the radial distance in terms of the dynamical symmetry, relating it indirectly to the matrix element of the dipole operator, without any further considerations. This violates condition b) above. We stress that the definition of the radial distance operator has to be independent of the Hamiltonian in the Hilbert space considered. The Hamiltonian determines whether there is a dynamical symmetry or not, which should be independent of the radial distance operator, while the Hilbert space as such is independent of the basis used.

The expectation value of the algebraic coordinate operator with respect to the coherent state is

$$\langle \mathbf{r}_m^a \rangle = \sqrt{\frac{2N\hbar}{m\omega}} \frac{\alpha_m}{(1 + \alpha^2)} = r_m^a \quad . \quad (14)$$

(Note that the α defined here is not the same as the α used in (6).)

We define this as the algebraic distance r_m^a , which is, by definition, of the order of one. Inverting this relation gives

$$\frac{\alpha_m}{(1 + \alpha^2)} = \sqrt{\frac{m\omega}{2N\hbar}} r_m^a \quad , \quad (15)$$

which again provides the dependence of α_m on N . This result suggests to redefine α_m in terms of δ_m and N as

$$\alpha_m = \frac{\delta_m}{\sqrt{N}} \quad (16)$$

which gives

$$\frac{\delta_m}{(1 + \frac{\delta^2}{N})} = \sqrt{\frac{m\omega}{2\hbar}} r_m^a \quad , \quad (17)$$

a dimensionless measure of the distance between the two nuclear clusters. For large N , the δ_m is directly proportional to r_m^a . We claim that this is a consistent way to define the separation between clusters through the variable δ_m .

The validity of (14), the definition of r_m^a , depends on the fluctuations of the related expectation value. The intercluster distance vector can always be chosen along the z axis. The square of the variation ($\langle \mathbf{r}_0^{a2} \rangle - \langle \mathbf{r}_0^a \rangle^2$) can then also be calculated, giving

$$\begin{aligned} \langle (\mathbf{r}_0^a)^2 \rangle - \langle \mathbf{r}_0^a \rangle^2 &= - \left(\frac{4\hbar}{m\omega} \right) \frac{\left(\frac{\delta_0^2}{N} \right)}{\left(1 + \frac{\delta^2}{N} \right)^2} + \left(\frac{\hbar}{2m\omega} \right) \frac{1 + \frac{\delta_0}{N}}{1 + \frac{\delta^2}{N}} \\ &\rightarrow \left(\frac{\hbar}{2m\omega} \right) \quad , \end{aligned} \quad (18)$$

also implying, in the second line, the large N limit. As long as the expectation value of the algebraic distance operator is greater than the square root of this expression, it is safe to identify the r_0^a as the distance between the two clusters. The square root of (18) gives numbers of the order of 1 fm. If r_m^a is of the order of the variation, it represents rather a *mean distance* of the two clusters.

The above considerations have been, up to now, relevant to the PACM, where no Pauli exclusion principle is taken into account. In the SACM some further differences appear. Because n_π has a lower bound, known as the Wildermuth condition [31], it was shown in [27] that for $\alpha = 0$ there is already a minimal distance between the clusters, given by $r_0 \sim \sqrt{n_0}$. This can be understood easily by noting that the maximal contribution of the radial wave function for non-zero oscillation quanta is located at $r > 0$. Equation (15) is the same, with the exception that r_m^a on the right hand side has to be substituted by $(r_m^a - r_{0,m})$, with $r_0^2 = \sum_m (-1)^m r_{0,m} r_{0,-m}$ being the square of the minimal distance r_0 of the two clusters [27]. Thus, the generally declared spherical limit, $\alpha = 0$, in the SACM corresponds to a minimal distance between the two clusters.

V. DISCUSSION

The question is now where the generally accepted procedure and our new proposal produce important differences. Focusing only at the potential is not always sufficient, because in the former procedure the n -body interactions are divided by powers in N , while in our new procedure these factors appear in the redefinition of the coherent state variable, e.g. α in the SACM and PACM. The final results look the same and the discussion on phase transitions give also the same results. Problems, however, appear when $N \rightarrow \infty$. When in the cluster model the one-body interaction is divided by N , e.g., $\hbar\omega \mathbf{n}_\pi \rightarrow \frac{\hbar\omega}{N} \mathbf{n}_\pi$ (note that $\hbar\omega$ is the distance in energy between two shells and is, therefore, fixed), for large N the excited states approach zero energy. If one does not

divide the one-body term by N , one has at least to divide the two-body interaction by $(N-1)$. Then, the argument repeats, i.e., for a pure π -boson two-body interaction the matrix element scale as $n_\pi^2/(N-1)$, which for $N \rightarrow \infty$ and $n_\pi \ll N$ approaches zero again.

With respect to the interaction parameters, the situation is different for a finite N , as in the IBA. Adjusting the parameters to experiment, one can increase the parameters of a two-body interaction such that they compensate for the $(N-1)$ in the denominator.

Therefore, the problem becomes only obvious in models like the SACM or PACM, where the total number of bosons represent only a cut-off, which has to approach infinity.

Another problem is related to the geometric values of observables, such as the quadrupole deformation (or the distance between two clusters). The principal argument is that the expectation value of an algebraic observable would be put equal to the corresponding geometrical observable, without using any additional factors. For example, if one determines the expectation value of the quadrupole operator Q_0 (or the algebraic distance operator r_m^a) with respect to a coherent state, its value scales approximately like $N\alpha^2$. If one associates to α a deformation, or distance, depending on the algebraic model, the value of Q_0 (r_m^a) would increase proportionally to N , i.e., it would be too large. In [9, 10] these expectation values are divided, without a clear justification, by N in order to get rid of the wrong N -dependence. In other words, the geometrical interpretation runs into severe problems, except when arbitrarily some N -dependent factors are introduced.

VI. CONCLUSIONS

In this contribution we discussed the geometrical mapping of algebraic models, using coherent states, and the interpretation of the coherent state variables. We showed that a consistent procedure involves the renormalization of the coherent state variables and not the many-body interaction parameters of the Hamiltonian.

This revised renormalization procedure introduces *in the geometrical mapping* the factor of $1/N$ for the 1-body interaction, $1/(N(N-1))$ for the 2-body interaction, etc. In conclusion, the renormalization procedure, as proposed in [13], introduces the N -dependent factors in the Hamiltonian, while the parameter α of the coherent state is taken as of the order of one, resulting in a semi-classical potential which does not depend on N for large values of N . In contrast, we leave the Hamiltonian untouched and the N -dependent factors enter via the renormalization procedure for α as presented here.

As already mentioned in the introduction, the structure of the model Hamiltonian depends on the system in consideration. In a two level model, with all states in the lower level occupied, one has to include N -dependent factors in order to get reasonable results. In other models, like the shell model and the ones presented here, the rules given in [12, 17] lead to inconsistencies and another renormalization procedure has to be applied. We also showed that for $N = \text{const.}$ the renormalization of the interaction parameters does not affect the results, provided that the parameters are free and adjusted to a specific system.

Not following the revised procedure leads to a wrong and inconsistent geometric interpretation and, at least, to conceptual difficulties.

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